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GRANULAR JUMPS DOWN AN INCLINED PLANE

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GRANULAR JUMPS DOWN AN INCLINED PLANE

M. C. SHEN*

Technical Summary Report #2529

June 1983

ABSTRACT

→ This document studies
~~We study~~ granular jumps generated in the gravity flow of granular materials down an inclined plane. The equations governing the granular flow are reduced to a sequence of boundary value problems of linear ordinary differential equations by means of an asymptotic method. Their solutions are used to determine the Burgers equation, which possesses a progressive wave solution to describe a smooth granular jump. The wave speed and a criterion for the stability of the granular jump are also obtained.

AMS (MOS) Subject Classifications: 73N99, 76A99

Key Words: Asymptotic method, granular jump, gravity flow, Burgers equation, stability

Work Unit Number 2 - Physical Mathematics

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SIGNIFICANCE AND EXPLANATION

Problems of flow of granular materials have attracted much attention recently because of their importance in industrial applications and geophysical situations. It is observed in laboratory tests that a granular jump may appear in the gravity flow of a granular material down an inclined plane. The granular jump may be considered as a wave front moving with constant velocity on the free surface of the granular material and connecting one uniform depth to another. The main contribution of this report is to develop an asymptotic method, which can describe the speed, wave profile and stability of the granular jump.

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GRANULAR JUMPS DOWN AN INCLINED PLANE

M. C. Shen^{*}

1. Introduction

In recent years there has been much interest in the study of the gravity flow of granular materials. Needless to say, granular flow problems occur frequently in industry as well as in nature such as handling of grains and sands, snow avalanches and rock falls. It is reported in laboratory tests (Savage, 1979) that smooth granular jumps may be observed when granular materials are moving down an inclined chute. As seen from loose snow avalanche or earth slide in nature, such a granular jump also seems to appear. On the other hand, since Goodman and Cowin (1972) proposed their continuum theory for flow of granular materials, only a few analytical results based upon their equations have been found, for example, steady state solutions by Goodman and Cowin (1971), and solutions of the linearized equations by Nunziato and Walsh (1977). The main purpose of this report, therefore, is to develop an asymptotic method for the study of the time-dependent granular flow under gravity down an inclined plane, which provides a means to describe the development of a smooth granular jump. Some results concerning granular jumps have been obtained by Morrison and Richmond (1976), and Savage (1979). However, their approach is based upon algebraic considerations of conservation laws for hydraulic jumps.

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In this report we adopt the equations formulated by Goodman and Cowin (1972) for the so-called cohesionless Coulomb granular material, and our approach used here is essentially an extension of the method we developed in a previous report (Shen, 1983) for the compressible viscous flow down an inclined channel. We briefly explain the method as follows. Assume that the granular flow down an inclined plane is two dimensional. We choose a coordinate system moving with the wave front of a granular jump so that the horizontal axis is parallel to the plane. A small parameter ϵ as the ratio of the vertical length scale H to the horizontal length scale L is introduced. We assume that the vertical velocity component is much smaller than the horizontal velocity component, and consider the granular flow for large time. Then we use ϵ to stretch various variables non-dimensionalized by appropriate units so that these assumptions are met. The solution of the governing equations is expanded in an asymptotic series in integral powers of ϵ , and the equations for the successive approximations determine the wave speed and an evolution equation for the free surface, which is the well-known Burgers equation. The signs of some coefficients in the Burgers equation determine whether the equation is ill-posed and provide a criterion for the stability of the flow.

The governing equations used here involve four material constants; two of them are related to the equilibrium stress and the other two appear in the dynamic stress. We first develop a general method in which all the material constants are assumed to be finite. However, the calculations of the wave speed and the coefficients in the Burgers equations are rather prohibitive. To simplify the derivations, we shall consider a special case, that is, we choose one of the material constants to be large in comparison with the others. In this case the equilibrium solution reduces to the one for the

incompressible viscous flow down an inclined plane, and the wave speed and the coefficients in the Burgers equation can be explicitly expressed in terms of the remaining constants. We also note that many other constitutive equations have been proposed for granular materials lately (Cowin and Satake, 1978; Savage, 1979; Nunziato, Passman and Thomas, 1980; Savage and Jeffrey, 1981; Sayed and Savage, 1983). Most of them are generalizations of the original formulation due to Goodman and Cowin (1972). The basic ideas developed here could also apply to these more general cases.

We formulate the problem for the general case in Section 2. The wave speed and the Burgers equation are derived in Section 3. In Section 4, we consider the special case. In Section 5, we define the criterion of stability, derive an expression for the wave profile of a granular jump and present some problems for further study.

2. Formulation

We consider a cohesionless Coulomb granular material flowing down a rough inclined plane so that we may impose the no-slip condition there. The flow is assumed to be two dimensional and a coordinate system moving at a speed σ^* in the x^* -direction is chosen so that the x^* -axis is parallel to the plane and the y^* -axis is positive upward (Figure 1). The governing equations are the following:

$$v_{t^*}^* + \nabla^* \cdot (v^* \bar{q}^*) = 0, \quad (1)$$

$$\gamma v^* (\bar{q}_{t^*}^* + \bar{q}^* \cdot \nabla^* \bar{q}^*) = \nabla^* \cdot \bar{T}^* + \gamma v^* \bar{q}^*, \quad (2)$$

subject to the boundary conditions:

At the free surface $\xi^* = y^* - \eta^*(x^*, t^*) = 0$,

$$\bar{T}^* \cdot \bar{n}^* = 0, \quad (3)$$

$$\bar{h}^* \cdot \bar{n}^* = 0, \quad (4)$$

$$\eta_{t^*}^* + u^* \eta_{x^*}^* - v^* = 0; \quad (5)$$

at the rough rigid plane,

$$\bar{q}^* = -\sigma^* \quad , \quad v^* = v_p^*. \quad (6)$$

Here $\nabla^* = (\partial/\partial x^*, \partial/\partial y^*) = (\partial/\partial x_1^*, \partial/\partial x_2^*)$, v^* is the volume distribution function, γ is the mass density assumed to be constant, $\bar{q}^* = (u^*, v^*) = (u_1^*, u_2^*)$ is the velocity of the flow, \bar{T}^* is the stress tensor defined by

$$T_{ij}^* = [-\beta^*(v^*)^2 + \alpha^*(\nabla^* v^*)^2 + 2\alpha^* v^* (\nabla^*)^2 v^*] \delta_{ij} - 2\alpha^* v_{x_i^*}^* v_{x_j^*}^* +$$

$$\lambda^* \nabla^* \cdot \bar{q}^* \delta_{ij} + 2\mu^* \sigma_{ij}^*,$$

$$\sigma_{ij}^* = (u_{ix_j^*}^* + u_{jx_i^*}^*)/2,$$

$\bar{g} = (g \sin \theta, -g \cos \theta)$ is the constant gravitational acceleration, θ is the angle of inclination of the plane to the horizontal assumed to be greater than the angle of internal friction but less than $\pi/2$, \bar{h}^* is the equilibrated stress vector given by $\bar{h}^* = 2\alpha^* \nabla^* v^*$, and \bar{n}^* is the vector

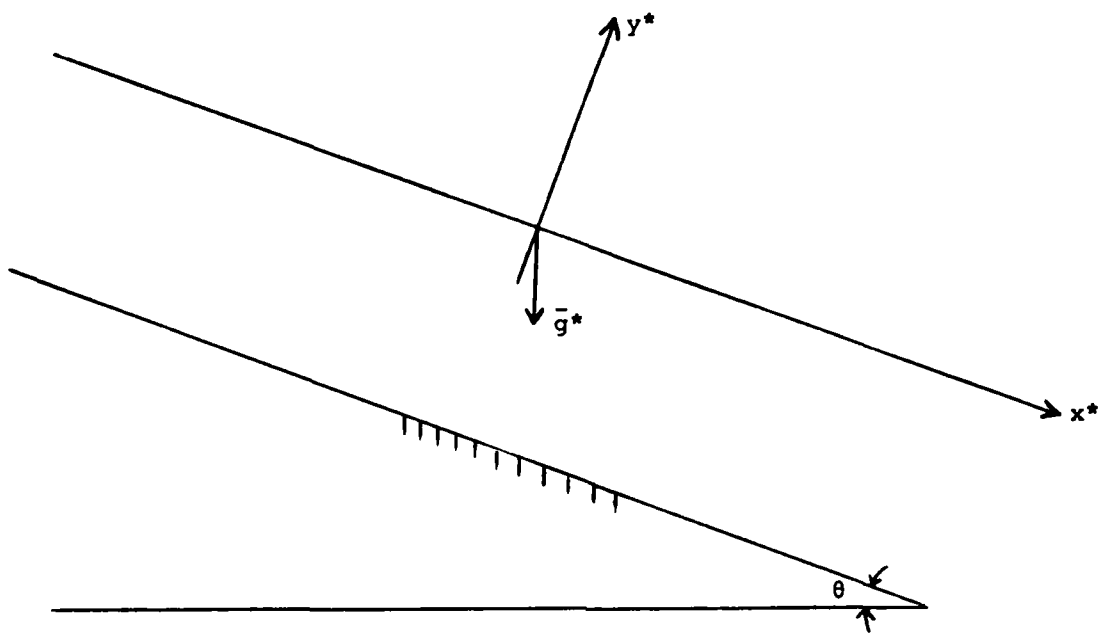


Figure 1. Coordinate system for the granular flow

$(\eta_{x^*}^*, -1)$ normal to the free surface. For simplicity, we assume that α^* , β^* , μ^* and ν_p^* are positive constants, and λ^* is also constant but satisfies $3\lambda^* + 2\mu^* > 0$.

We introduce the following nondimensional variables:

$$v = v^*/\nu_p^*, (u, v) = (u^*, \epsilon v^*)(gH)^{-1/2},$$

$$t = \epsilon^{-2} t^*(H/g)^{-1/2}, (x, y) = (\epsilon^{-1} x^*, y^*)H^{-1},$$

$$\sigma = \sigma^*(gH)^{-1}, \beta = \beta^*(\gamma \nu_p^* gH)^{-1},$$

$$\alpha = \alpha^*(\gamma \Delta g H^3)^{-1}, (\lambda, \mu) = (\lambda^*, \mu^*)(\gamma \nu_p^* (gH)^{1/2} H)^{-1}$$

$$\epsilon = H/L.$$

In terms of them, (1) to (6) become

$$\epsilon v_t + (vu)_x + (vv)_y = 0, \quad (7)$$

$$\begin{aligned} \epsilon v(\epsilon^2 u_t + uu_x + vu_y) = -2\beta \epsilon v v_x + 2\alpha v(\epsilon^2 v_{xx} + v_{yy})_x \\ + (\lambda + \mu)\epsilon^2(u_x + v_y)_x + \mu(\epsilon^2 u_{xx} + u_{yy}) + v \sin \theta, \end{aligned} \quad (8)$$

$$\begin{aligned} \epsilon^2 v(\epsilon v_t + uv_x + vv_y) = -2\beta v v_y + 2\alpha v(\epsilon^2 v_{xx} + v_{yy})_y \\ + (\lambda + \mu)\epsilon(u_x + v_y)_y + \mu\epsilon(\epsilon^2 v_{xx} + v_{yy}) - v \cos \theta; \end{aligned} \quad (9)$$

at $y = \eta(x, t)$,

$$\begin{aligned} [-\beta v^2 + \alpha(\epsilon^2 v_x^2 + v_y^2) + 2\alpha v(\epsilon^2 v_{xx} + v_{yy}) - 2\alpha \epsilon^2 (v_x)^2 \\ + \lambda \epsilon(u_x + v_y) + 2\mu \epsilon u_x] \epsilon \eta_x + 2\alpha \epsilon v v_y - \mu(u_y + \epsilon^2 v_x) = 0, \end{aligned} \quad (10)$$

$$[-2\alpha \epsilon v v_y + \mu(u_y + \epsilon^2 v_x)] \epsilon \eta_x + \beta v^2 - \alpha(\epsilon^2 v_x^2 + v_y^2) \quad (11)$$

$$-2\alpha v(\epsilon^2 v_{xx} + v_{yy}) + 2\alpha (v_y)^2 - \lambda \epsilon(u_x + v_y) - 2\mu \epsilon v_y = 0,$$

$$\epsilon^2 \eta_x v_x - v_y = 0, \quad (12)$$

$$\epsilon \eta_t + u \eta_x - v = 0 \quad ; \quad (13)$$

at $y = -1$,

$$u = -\sigma, \quad v = 0, \quad \psi = 1 \quad . \quad (14)$$

3. Wave speed and the Burgers equation

Assume that u, v, η and σ possess an asymptotic expansion of the form

$$\phi = \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \dots \quad (15)$$

Without loss of generality, we may assume $\sigma = \sigma_0 + \epsilon \sigma_1$. Substitution of (15) in (7) to (14) yields a sequence of equations and boundary conditions for the successive approximations. The equations for the zeroth approximation are

$$\mu u_{0yy} = -v_0 \sin \theta, \quad (16)$$

$$2\alpha v_{0yyy} - 2\beta v_{0y} = \cos \theta; \quad (17)$$

at $y = 0$,

$$u_{0y} = 0, \quad (18)$$

$$2\alpha v_{0yy} - \beta v_0 = 0, \quad (19)$$

$$v_{0y} = 0; \quad (20)$$

at $y = -1$,

$$u_0 = -\sigma_0, \quad v_0 = 1 \quad (21)$$

where u_0 is assumed to be a function of y only and $v_0 = \eta_0 = 0$. The solutions of (16) to (21) for u_0 and v_0 can be easily found, and a discussion of these solutions is given in Goodman and Cowin (1971). Following them, we define

$$l = (\beta/\alpha)^{1/2} \quad \text{and} \quad M = \cos \theta / (2\beta), \quad (22)$$

and obtain

$$v_0 = Ml^{-1} \sinh ly + (1 - M + Ml^{-1} \sinh l)(1 + \cosh l)^{-1}(1 + \cosh ly) - My, \quad (23)$$

$$\begin{aligned} u_0 &= \phi_0 - \sigma_0 \\ &= \mu^{-1} \sin \theta \{-Ml^{-3}(\sinh l + \sinh ly) + (1 - M + Ml^{-1} \sinh l)x \\ &\quad (1 + \cosh l)^{-1}[(1 - y^2)/2 + l^{-2}(\cosh l - \cosh ly)] + (M/6)(1 + y^3) \\ &\quad + Ml^{-2}(1 + y)\} - \sigma_0. \end{aligned} \quad (24)$$

For small ℓ , we may expand v_0 and ϕ_0 in integral powers of ℓ and obtain

$$v_0 \simeq 1 + \ell^2 [(M/6)y^3 + y^2/4 + (M/6 - 1/4)] + \dots, \quad (25)$$

$$\begin{aligned} \phi_0 \simeq \mu^{-1} \sin \theta \{ (1-y^2)/2 + \ell^2 [(1-y^2)(M/12 - 1/8) + (1-y^4)/48 \\ - M(1+y^5)/120] + \dots \}. \end{aligned} \quad (26)$$

We see from (25) and (26) that as $\ell \rightarrow 0$, v_0 , u_0 tend to the solutions for an incompressible viscous flow fown an inclined plane (Landau and Lifshitz, 1959). This result motivates our choice of α as a large parameter in the next section so that expressions for the wave speed and the coefficients in the Burgers equation can be greatly simplified. On the other hand, if $\ell \rightarrow \infty$, v_0 tends to a discontinuous solution when $M \neq 1$ and becomes zero at $y = 0$ (Goodman and Cowin, 1971). Therefore, we shall not study this case in the sequel.

The equations for the first approximation are

$$(v_1 u_0 + v_0 u_1)_x + (v_0 v_1)_y = 0, \quad (27)$$

$$\mu u_{1yy} = -v_1 \sin \theta, \quad (28)$$

$$\alpha v_{1yyy} - \beta v_{1y} = 0; \quad (29)$$

at $y = 0$,

$$u_{1y} + u_{0yy} \eta_1 = 0, \quad (30)$$

$$\alpha(v_0 v_{1yy} + v_{0yy} v_1) - \beta v_0 v_1 = -\alpha v_0 v_{0yyy} \eta_1, \quad (31)$$

$$v_{1y} + v_{0yy} \eta_1 = 0; \quad (32)$$

$$u_0 \eta_{1x} - v_1 = 0 \quad (33)$$

at $y = -1$,

$$u_1 = -\sigma_1, \quad v_1 = 0, \quad v_{1y} = 0. \quad (34)$$

As observed from (28) to (32) and (34), we may express v_1 and u_1 as

$$v_1 = \psi_1 \eta_1, \quad u_1 = \phi_1 \eta_1 - \sigma_1, \quad (35)$$

at $y = 0$,

$$\begin{aligned} & (-\beta v_0^2 + 2\alpha v_0 v_{0yy}) \eta_{1x} - \mu (u_{2y} + u_{1y} \eta_1 + u_{0yy} \eta_2 + u_{0yyy} \eta_1^2/2) = 0 \\ & \beta (2v_0 v_{0yy} \eta_1^2/2 + v_1^2 + 2v_0 v_{1y} \eta_1 + 2v_0 v_2) + 2[(v_{0yy})^2 \eta_1^2 + (v_{1y} \\ & + 2v_{0yy} v_{1y} \eta_1] - 2\alpha [v_0 (v_{0yyy} \eta_2 + v_{0yyyy} \eta_1^2/2 + v_{1yyy} \eta_1 + v_2 \\ & + v_{0yy} (v_{0yy} \eta_1^2/2 + v_{1y} \eta_1 + v_2 + v_1 v_{1yy}) + v_1 (v_{0yyy} \eta_1 + v_{1y} \\ & \lambda (u_{1x} + v_{1y}) - 2\mu v_{1y} = 0 \end{aligned}$$

$$v_{2y} + v_{1y} \eta_1 + v_{0yy} \eta_1^2/2 = 0$$

$$\eta_{1t} + u_0 \eta_{2x} + u_1 \eta_{1x} + u_{0y} \eta_1 \eta_{1x} - v_2 - v_{1y} \eta_1 = 0$$

at $y = -1$,

$$u_2 = v_2 = v_2 = 0$$

From (17), (19), (24), (35), (36) and (42), we may rewrite (and (53) as follows:

$$\begin{aligned} L_1(u_2) = \mu u_{2yy} = & -v_2 \sin \theta + [v_0 (\phi_0 - \sigma_0) \phi_1 + 2\beta v_0 \psi_1 - 2\alpha v \\ & + v_0 f_1 \phi_{0y}] \eta_{1x} \end{aligned}$$

$$L_2(v_2) = 2\alpha v_0 v_{2yyy} - \alpha \beta v_0 v_{2y} = -[(\lambda + \mu)(\phi_{1y} + f_{1yy}) + \mu f_{1yy}]$$

at $y = 0$,

$$R_1(u_2) = \mu u_{2y} = (-\beta v_0^2 + 2\alpha v_0 v_{0yy}) \eta_{1x} - \mu (\phi_{1y} + \phi_{0yyy}/2) \eta_1^2 - \mu \phi_{0yy}$$

$$B_2(v_2) = 2\alpha v_0 v_{2yy} - \beta v_0 v_2 = [\lambda (\phi_1 + f_{1y}) - 2\mu f_{1y}] \eta_{1x}$$

$$+ \{\beta (v_0 v_{0yy} + \psi_1^2 + 2v_0 \psi_{1y}) - 2\alpha [(v_{0yy})^2/2 - (\psi_{1y})^2/2 + v_0] +$$

$$\psi_{1yyy}) + v_{0yy} \psi_{1y} \psi_{1yy} + \psi_1 (v_{0yyy} + \psi_{1yy})\} \eta_1^2$$

$$- 2\alpha v_0 v_{0yyy} \eta_2$$

$$B_3(v_2) = v_{2y} = -(\psi_{1y} + v_{0yy}/2)n_1^2, \quad (58)$$

at $y = -1$,

$$u_2 = v_2 = 0. \quad (59)$$

It is observed from (54) to (59) that u_2 and v_2 can be expressed in the following forms

$$u_2 = \phi_2 n_{1x} + \phi_3 n_1^2 + \phi_4 n_2, \quad (60)$$

$$v_2 = \psi_2 n_{1x} + \psi_3 n_1^2 + \psi_4 n_2. \quad (61)$$

Here ψ_2 to ψ_4 and ϕ_2 to ϕ_4 are functions of y and satisfy

$$L_2(\psi_i) = F_i, \quad i = 2, 3, 4, \quad (62)$$

$$B_2(\psi_i) = G_i, \quad B_3(\psi_i) = H_i, \quad \text{at } y = 0, \quad (63)$$

$$\psi_i = 0 \quad \text{at } y = -1; \quad (64)$$

$$L_1(\phi_j) = I_j, \quad j = 2, 3, 4, \quad (65)$$

$$B_1(\phi_j) = J_j \quad \text{at } y = 0, \quad (66)$$

$$\phi_j = 0 \quad \text{at } y = -1, \quad (67)$$

where

$$F_2 = -(\lambda + \mu)(\phi_{1y} + f_{1yy}) - \mu f_{1yy},$$

$$F_i = 0, \quad i = 3, 4,$$

$$G_2 = \lambda(\phi_1 + f_{1y}) - 2\mu f_{1y},$$

$$G_3 = \beta(v_0 v_{0yy} + \psi_1^2 + 2v_0 \psi_{1y}) - 2\alpha[(v_{0yy})^2/2 - (\psi_{1y})^2/2 + v_0(v_{0yyyy}/2 + \psi_{1yyy}) + v_{0yy}\psi_{1y}\psi_{1yy} + \psi_1(v_{0yyy} + \psi_{1yy})],$$

$$G_4 = -2\alpha v_0 v_{0yyy},$$

$$H_3 = -(\psi_{1y} + v_{0yy}/2), \quad H_i = 0, \quad i = 3, 4,$$

$$I_2 = -\psi_2 \sin \theta + v_0(\phi_0 - \sigma_0)\phi_1 + 2\beta v_0 \psi_1 - 2\alpha v_0 \psi_{1yy} + v_0 f_1 \phi_{0y}, \quad I_3 = -\psi_3 \sin \theta, \quad I_4 = -\psi_4 \sin \theta,$$

$$J_1 = -\beta v_0^2 + 2\alpha v_0 v_{0yy}, \quad J_2 = 0, \quad J_3 = -\mu(\phi_{1y} + \phi_{0yyy}/2),$$

$$J_4 = -\mu \phi_{0yy} \quad .$$

In principle, we can solve (62) to (67) for ψ_i and ϕ_j , and in turn u_2 and v_2 can be determined from (60), (61). Now we are in a position to determine the Burgers equation. We integrate (46) with respect to y from $y = -1$ to $y = 0$ and make use of (18), (33) to (35), (52) to (53), (60) and (61), to obtain

$$\begin{aligned} & \int_{-1}^0 \psi_1 dy \, n_{1t} + \int_{-1}^0 [(\psi_2 n_{1x} + \psi_3 n_1^2 + \psi_4 n_2)(\phi_0 - \sigma_0) + \psi_1(\phi_1 n_1 - \sigma_1) n_1 \\ & + v_0(\phi_2 n_{1x} + \phi_3 n_1^2 + \phi_4 n_2)]_x dy + \psi_1(0)[\phi_0(0) - \sigma_0] n_1 n_{1x} \\ & + n_{1t} + (\phi_0 - \sigma_0) n_{2x} + (\phi_1 n_1 - \sigma_1) n_{1x} + -f_{1y}(0) n_1 n_{1x} = 0 \quad . \end{aligned} \quad (68)$$

Note that the coefficient of n_2 vanishes because of (45). By rearranging the terms, we finally obtain the Burgers equation

$$m_0 n_{1t} + m_1 n_{1x} + m_2 n_1 n_{1x} = m_3 n_{1xx} \quad , \quad (69)$$

Which is our main result. Here

$$m_0 = \int_{-1}^0 \psi_1 dy + 1 \quad , \quad m_1 = -\sigma_1 m_0 \quad ,$$

$$m_2 = 2 \int_{-1}^0 [\psi_3(\phi_0 - \sigma_0) + \psi_1 \phi_1 + v_0 \phi_3] dy + \psi_1(0)[\phi_0(0) - \sigma_0] + \phi_1(0) - f_{1y}(0) \quad ,$$

$$m_3 = - \int_{-1}^0 [\psi_2(\phi_0 - \sigma_0) + v_0 \phi_2] dy \quad .$$

(69) can be used to study the development of a granular jump.

4. A special case

In Section 3, we derived the expressions for the wave speed σ_0 and the coefficients for the Burgers equation. However, they depend upon θ , α , β , λ and μ , and are too complicated for applications. Therefore, as motivated by the discussion of v_0 and ϕ_0 given in the last Section, we assume $\alpha = 1/\epsilon$ to be a large parameter. Substitution of (15) in (8) to (14) also yields a sequence of equations and bounding conditions. The equations for the zeroth approximation are the following:

$$\mu u_{0yy} = -v_0 \sin \theta, \quad (70)$$

$$v_{0yyy} = 0; \quad (71)$$

at $y = 0$,

$$u_{0y} = 0, \quad (72)$$

$$v_{0yy} = 0, \quad v_{0y} = 0; \quad (73)$$

at $y = -1$,

$$u_0 = -\sigma_0, \quad v_0 = 1. \quad (74)$$

It is easily seen that

$$v_0 = 1, \quad (75)$$

$$u_0 = [\sin \theta / (2\mu)] (1 - y^2) - \sigma_0. \quad (76)$$

(75) and (76) are solutions for the incompressible viscous flow down an inclined plane.

The equations for the first approximation are

$$(v_1 u_0 + v_0 u_1)_x + (v_0 v_1)_y = 0, \quad (77)$$

$$\mu u_{1yy} = -2v_{1yyx} - v_1 \sin \theta; \quad (78)$$

$$2v_{1yyy} = \cos \theta, \quad (79)$$

at $y = 0$,

$$u_{1y} + u_{0yy} \eta_1 = 0, \quad (80)$$

$$v_{1yy} = 0, \quad (81)$$

$$v_{1y} = 0, \quad (82)$$

$$u_0 \eta_{1x} - v_1 = 0; \quad (83)$$

at $y = -1$,

$$u_1 = -\sigma_1, \quad v_1 = v_1 = 0. \quad (84)$$

It is found from (78) to (82), and (84) that

$$v_1 = (\cos \theta/12)(1+y^3) \quad (85)$$

$$u_1 = [\sin \theta \cos \theta/(2\psi\mu)](9/10 - y^2 - y^5/10) - \sigma_1 + (\sin \theta/\mu)(1+y)\eta_1. \quad (86)$$

We now integrate (77) with respect to y from $y = -1$ to y and obtain, by (75), (76), (85) and (86),

$$v_1 = -\int_{-1}^y u_{1x} dy = -[\sin \theta/(2\mu)](1+y)^2 \eta_{1x}. \quad (87)$$

It follows from (76), (83) and (87), that

$$[\sin \theta/(2\mu) - \sigma_0] \eta_{1x} = -\sin \theta/(2\mu) \eta_{1x}.$$

Assume that $\eta_{1x} \neq 0$ and we have

$$\sigma_0 = \sin \theta/\mu. \quad (88)$$

Now we proceed to the equations for the second approximation

$$v_{1t} + (v_2 u_0 + v_1 u_1 + v_0 u_2)_x + (v_1 v_1 + v_0 v_2)_y = 0, \quad (89)$$

$$u_0 u_{1x} + v_1 u_{0y} = 2v_{2yyx} + \mu u_{2yy} + v_2 \sin \theta, \quad (90)$$

$$-2\beta v_{1y} + 2(v_{2yyy} + v_1 v_{1yyy}) - v_1 \cos \theta = 0; \quad (91)$$

at $y = 0$,

$$(-\beta + 2v_{1yy}) \eta_{1x} - \mu(u_{2y} + u_{1y} \eta_1 + u_{0yy} \eta_2 + u_{0yyy} \eta_1^2/2) = 0, \quad (92)$$

$$2\beta v_1 = (v_{1y})^2 - 2(v_{2yy} - v_1 v_{1yy} + v_{1yyy} \eta_1) = 0, \quad (93)$$

$$v_{2y} + v_{1yy} \eta_1 = 0, \quad (94)$$

$$\eta_{1t} + u_0 \eta_{2x} + u_1 \eta_{1x} + u_{0y} \eta_1 \eta_{1x} - v_2 - v_{1y} \eta_1 = 0; \quad (95)$$

at $y = -1$,

$$u_2 = v_2 = v_2 = 0. \quad (96)$$

From (85), (91), (93), (94) and (96), we find that

$$v_2 = (\beta \cos \theta/24)(-9/10 + y^2 + y^5/10) + (\cos \theta/4)(1-y^2) \eta_1. \quad (97)$$

Then from (76), (86), (87), (90), (92), (96) and (97) it is obtained that

$$\begin{aligned} \mu u_2 = & [-\beta - \cos \theta/2 - \sin^2 \theta/(8\mu^2) - \beta y + y^2 \cos \theta/2 + \sin^2 \theta(-y^3/6 + y^4/12)]\eta_{1x} \\ & + \cos \theta \sin \theta(5-6y^2+y^4)\eta_1/48 - \sin \theta(1+y)\eta_1^2 + \sin \theta(1+y)\eta_2 \\ & - \beta \cos \theta \sin \theta(155-189y^2 + 35y^4 + y^7)/10080 . \end{aligned} \quad (98)$$

Making use of (74), (76), (83) to (88) and (95) to (98), we integrate (89)

with respect to y from $y = -1$ to $y = 0$ to obtain

$$\eta_{1t} + m_1 \eta_{1x} + m_2 \eta_1 \eta_{1x} = m_3 \eta_{1xx} , \quad (99)$$

where the coefficient of η_2 vanishes because of (88), and

$$m_1 = \cos \theta \sin \theta/(24\mu) - \sigma_1 ,$$

$$m_2 = \sin \theta/\mu ,$$

$$m_3 = \beta/2 + \cos \theta/3 + 23 \sin^2 \theta/(240\mu^2) .$$

5. Discussion

It is well known that the Burgers equation as given by (69) or (99) is ill-posed if the coefficients m_0 and m_3 are of opposite signs. Therefore, we may use this result to define the criterion of stability. Since in (69), m_0 and m_3 depend upon $\theta, \alpha, \beta, \lambda$ and μ , we say a granular jump is stable if m_0 and m_3 are of same sign i.e. $m_3/m_0 = f(\theta, \alpha, \beta, \lambda, \mu) > 0$. For (99), $m_0 = 1$ and $m_3 > 0$. Hence a granular jump is always stable in this sense.

The solution method for the Burgers equation is also well known and we only give a brief discussion of the steady state solutions of (69) and (99) in reference to a moving frame. Assume all the coefficients in (69) are not zero, $m_3/m_0 > 0$ and $\eta_1, \eta_{1x} \rightarrow 0$ as $x \rightarrow \infty$. It is easily found that

$$\eta_1 = -(m_1/m_2)[1 + \tanh(m_1(x+c)/(2m_3))]$$

where c is an arbitrary constant and we require $m_1/m_3 < 0$. Since in (69), $m_1 = -m_0\sigma_1$, $m_3/m_0 > 0$ and $m_1/m_3 < 0$ imply $\sigma_1 > 0$. Since $\sigma = \sigma_0 + \epsilon\sigma_1 > \sigma_0$, we say the granular jump moves at a supercritical speed. If $m_0/m_2 > 0$, then $\eta_1 \rightarrow 2m_0\sigma_1/m_2 > 0$ as $x \rightarrow -\infty$. In this case η_1 decreases smoothly from $2m_0\sigma_1/m_2$ to zero as x increases. On the other hand, if $m_0/m_2 < 0$, then η_1 increases smoothly from $\omega_0\sigma_1/m_2 < 0$ to zero as x increases. For the special case considered in Section 4, $m_0 = 1$, $m_2 > 0$, $m_3 > 0$. Hence $m_1/m_3 < 0$ implies $\sigma_1 > \cos\theta \sin\theta/(24\mu) > 0$ and the speed of the granular jump is supercritical. Furthermore, $m_2 > 0$ implies that we always have a decreasing granular jump.

Finally we consider some problems for further study. If the free surface of the granular material is replaced by a rigid plane, then one boundary condition is lost and we do not have enough boundary conditions to determine the flow. In fact, in the gravity flow of a granular material between two vertical plates (Goodman and Cowin, 1971), a slug flow appears in the central

part of the vertical channel. It should be of interest to extend the approach to the study of the granular flow between two parallel inclined planes. Furthermore, if $m_0/m_3 < 0$, an investigation of the ill-posed problem is certainly of importance. Finally the asymptotic method used here is based upon the existence of a solution given by (15). A justification of the method, needless to say, will be a significant contribution to granular flow problems.

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